

ELLIPTIC OPERATORS WITH UNBOUNDED DIFFUSION COEFFICIENTS PERTURBED BY INVERSE SQUARE POTENTIALS IN L^p -SPACES

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ABSTRACT. In this paper we give sufficient conditions on $\alpha \geq 0$ and $c \in \mathbb{R}$ ensuring that the space of test functions $C_c^\infty(\mathbb{R}^N)$ is a core for the operator

$$L_0 u = (1 + |x|^\alpha) \Delta u + \frac{c}{|x|^2} u =: Lu + \frac{c}{|x|^2} u,$$

and L_0 with a suitable domain generates a quasi-contractive and positivity preserving C_0 -semigroup in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. The proofs are based on some L^p -weighted Hardy's inequality and perturbation techniques.

1. INTRODUCTION

Let us consider the elliptic operator

$$L = (1 + |x|^\alpha) \Delta,$$

where $\alpha \geq 0$. In this paper we want to study the perturbation of L with a singular potential. More precisely, we consider the operator

$$L_0 = L + \frac{c}{|x|^2}$$

and we look for optimal conditions on $c \in \mathbb{R}$ and α ensuring that L_0 with a suitable domain generates a positivity preserving C_0 -semigroup in $L^p(\mathbb{R}^N)$.

Let us recall first some known results for Schrödinger operators with inverse-square potentials.

It is known, see [19, Theorem 2], that the realization A_2 of the Schrödinger operator $\mathcal{A} = \Delta + c|x|^{-2}$ in $L^2(\mathbb{R}^N)$ is essentially selfadjoint on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ if and only if

$$c \leq \frac{(N-2)^2}{4} - 1 =: c_0,$$

cf. [4, Proposition VII.4.1] or [18, Theorem X.11], when $N \geq 5$.

The characterization of the existence of positive weak solutions to the parabolic problem associated with the operator \mathcal{A} was first discovered by Baras and Goldstein [2], where they proved that a positive weak solution exists if and only if $c \leq \frac{(N-2)^2}{4}$.

Using perturbation techniques it is proved in [15, Theorem 6.8] that A_2 is self-adjoint provided that $c < c_0$. These techniques were generalized to the L^p -setting, $1 < p < \infty$, and it is obtained that A_p , the realization of \mathcal{A} in $L^p(\mathbb{R}^N)$, with domain $W^{2,p}(\mathbb{R}^N)$ generates a contractive and positive C_0 -semigroup in $L^p(\mathbb{R}^N)$, and $C_c^\infty(\mathbb{R}^N)$ is a core for A_p , if $N > 2p$ and

$$c < \frac{(p-1)(N-2p)N}{p^2} =: \beta_0,$$

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see [16, Theorem 3.11]. In the case where $N \leq 2p$, it is proved that A_p with domain $D(A_p) = W^{2,p}(\mathbb{R}^N) \cap \{u \in L^p(\mathbb{R}^N); |x|^{-2}u \in L^p(\mathbb{R}^N)\}$ is m -sectorial if $c < \beta_0$, see [16, Theorem 3.6].

If one replaces the Laplacian by the Ornstein-Uhlenbeck operator similar results were obtained recently in [3, 7].

In this paper we obtain similar results as in [16, Theorem 3.11] when replacing Δ by L . We discuss also the generation of a C_0 -semigroup of the operator $(1 + |x|^\alpha)\Delta - \eta|x|^\beta + \frac{c}{|x|^2}$, where η is a positive constant, $\alpha \geq 2$ and $\beta > \alpha - 2$.

Now, let us recall some definitions. An operator $(A, D(A))$ on a Banach space X is called accretive if $-A$ is dissipative. It is m -accretive if A is accretive and $X = R(\lambda + A)$, the range of the operator $(\lambda + A)$. An accretive operator $(A, D(A))$ is called essentially m -accretive if its closure \bar{A} is m -accretive.

Our approach relies on the following perturbation result due to N. Okazawa, see [16, Theorem 1.7].

Theorem 1.1. *Let A and B be linear m -accretive operators in $L^p(\mathbb{R}^N)$, with $p \in (1, +\infty)$. Let D be a core of A . Assume that*

- (i) *there are constants $\tilde{c}, a \geq 0$ and $k_1 > 0$ such that for all $u \in D$ and $\varepsilon > 0$*

$$\operatorname{Re}\langle Au, \|B_\varepsilon u\|_p^{2-p} |B_\varepsilon u|^{p-2} B_\varepsilon u \rangle \geq k_1 \|B_\varepsilon u\|_p^2 - \tilde{c} \|u\|_p^2 - a \|B_\varepsilon u\|_p \|u\|_p$$

where B_ε denote the Yosida approximation of B ;

- (ii) *$\operatorname{Re}\langle u, \|B_\varepsilon u\|_p^{2-p} |B_\varepsilon u|^{p-2} B_\varepsilon u \rangle \geq 0$, for all $u \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$;*
- (iii) *there is $k_2 > 0$ such that $A - k_2 B$ is accretive.*

Set $k = \min\{k_1, k_2\}$. If $c > -k$ then $A + cB$ with domain $D(A + cB) = D(A)$ is m -accretive and any core of A is also a core for $A + cB$. Furthermore, $A - kB$ is essentially m -accretive on $D(A)$.

In order to apply the above theorem, we need some preliminary results on the operator L and some Hardy's inequalities.

2. PRELIMINARY RESULTS

Let us begin with the generation results for suitable realizations L_p of the operator L in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. Such results have been proved in [6, 9, 11]. More specifically, the case $\alpha \leq 2$ has been investigated in [6] for $1 < \alpha \leq 2$ and in [9] for $\alpha \leq 1$, where the authors proved the following result.

Theorem 2.1. *If $\alpha \in [0, 2]$ then, for any $p \in (1, +\infty)$, the realization L_p of L with domain*

$$D_p = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^\alpha |D^2 u|, |x|^{\alpha/2} |\nabla u| \in L^p(\mathbb{R}^N)\}$$

generates a positive and strongly continuous analytic semigroup. Moreover $C_c^\infty(\mathbb{R}^N)$ is a core for L_p .

The case $\alpha > 2$ is more involved and is studied in [11], where the following facts are established.

Theorem 2.2. *Assume that $\alpha > 2$.*

1. *If $N = 1, 2$, no realization of L in $L^p(\mathbb{R}^N)$ generates a strongly continuous (resp. analytic) semigroup.*
2. *The same happens if $N \geq 3$ and $p \leq N/(N-2)$.*
3. *If $N \geq 3$, $p > N/(N-2)$ and $2 < \alpha \leq (p-1)(N-2)$, then the maximal realization L_p of the operator L in $L^p(\mathbb{R}^N)$ with the maximal domain*

$$D_{\max} = \{u \in W^{2,p}(\mathbb{R}^N) : (1 + |x|^\alpha)\Delta u \in L^p(\mathbb{R}^N)\}$$

generates a positive C_0 -semigroup of contractions, which is also analytic if $\alpha < (p-1)(N-2)$.

4. If $N \geq 3$, $p > N/(N-2)$ and $2 < \alpha < \frac{N(p-1)}{p}$ the domain D_{\max} coincides with the space

$$\widehat{D}_p = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^{\alpha-2}u, |x|^{\alpha-1}|\nabla u|, |x|^\alpha|D^2u| \in L^p(\mathbb{R}^N)\}.$$

Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for L .

If we consider the operator $\tilde{L} := L - \eta|x|^\beta$ with $\eta > 0$ and $\beta > \alpha - 2$ then we can drop the above conditions on p , α and N , as the following result shows, see [1], where the quasi-contractivity can be deduced from the proof of Theorem 4.5 in [1].

Theorem 2.3. *Assume $N \geq 3$. If $\alpha > 2$ and $\beta > \alpha - 2$ then, for any $p \in (1, \infty)$, the realization \tilde{L}_p of \tilde{L} with domain*

$$\widetilde{D}_p = \{u \in W^{2,p}(\mathbb{R}^N) : |x|^\beta u, |x|^{\alpha-1}|\nabla u|, |x|^\alpha|D^2u| \in L^p(\mathbb{R}^N)\}$$

generates a positive and strongly continuous quasi-contractive analytic semigroup. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for \tilde{L}_p .

From now on we assume $N \geq 3$, $\alpha \geq 0$. We set

$$\gamma_\alpha = \left(\frac{N + \alpha - 2}{p} \right)^2 \quad (2.1)$$

and recall the following Hardy's inequality. For a proof we refer to [16, Lemma 2.2 & Lemma 2.3] for the case $\alpha = 0$ and [11, Appendix] for $\alpha \geq 2$. Here we give a simple proof based on the method of vector fields introduced by Mitidieri in [13], which holds for any $\alpha \geq 0$.

Lemma 2.4. *For every $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, one has*

$$\gamma_\alpha \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} |x|^\alpha dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} |x|^\alpha dx. \quad (2.2)$$

The inequality holds true even if u is replaced by $|u|$.

Proof. By density, it suffices to prove (2.2) for $u \in C_c^1(\mathbb{R}^N)$. So, for every $\lambda \geq 0$, let us consider the vector field $F(x) = \lambda \frac{x}{|x|^2} |x|^\alpha$, $x \neq 0$, and set $d\mu(x) = |x|^\alpha dx$. Integrating by parts and applying Hölder and Young's inequalities we get

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^p \operatorname{div} F dx &= \lambda(N-2+\alpha) \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu \\ &= -p\lambda \int_{\mathbb{R}^N} |u|^{p-2} \mathcal{R}e(u \nabla \bar{u}) \cdot \frac{x}{|x|^2} d\mu \\ &\leq p\lambda \left(\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} d\mu \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu \right)^{\frac{1}{2}} \\ &\leq \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} d\mu + \frac{\lambda^2 p^2}{4} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} d\mu. \end{aligned}$$

In the computations above, we used the identity $\nabla |u|^p = p|u|^{p-2} \mathcal{R}e(u \nabla \bar{u})$. Hence,

$$\left[\lambda(N-2+\alpha) - \frac{\lambda^2 p^2}{4} \right] \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} |x|^\alpha dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{p-2} |x|^\alpha dx.$$

By taking the maximum over λ of the function $\psi(\lambda) = \lambda(N-2+\alpha) - \lambda^2 p^2/4$, we get (2.2).

We note here that the integration by parts is straightforward when $p \geq 2$. For $1 < p < 2$, $|u|^{p-2}$ becomes singular near the zeros of u . Also in this case the integration by parts is allowed, see [10].

By using the identity $\nabla |u|^p = p|u|^{p-1} \nabla |u|$ in the computations above, the statement holds with u replaced by $|u|$. \square

Remark 2.5. The constant γ_α in (2.2) is optimal, as shown in [11, Appendix].

Remark 2.6. Hardy's inequality (2.2) holds even if u is replaced by $u_+ := \sup(u, 0)$, since $u_+ \in W^{1,p}(\mathbb{R}^N)$, whenever $u \in W^{1,p}(\mathbb{R}^N)$ (cf. [8, Lemma 7.6]).

As a consequence of Lemma 2.4 we have the following results.

Proposition 2.7. *Assume $\alpha \leq (N-2)(p-1)$. Let $V \in L^p_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$. If $V(x) \leq \frac{c}{|x|^2}$, $x \neq 0$, with $c \leq (p-1)\gamma_0$, then $L+V$ with domain $C_c^\infty(\mathbb{R}^N)$ is dissipative in $L^p(\mathbb{R}^N)$.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. Take $\delta > 0$ if $1 < p < 2$ and $\delta = 0$ if $p \geq 2$. Then we have

$$\begin{aligned} \langle Lu, u(|u|^2 + \delta)^{\frac{p-2}{2}} \rangle &= - \int_{\mathbb{R}^N} \nabla u \cdot \nabla \left(\bar{u}(|u|^2 + \delta)^{\frac{p-2}{2}} \right) (1 + |x|^\alpha) dx \\ &\quad - \alpha \int_{\mathbb{R}^N} \bar{u}(|u|^2 + \delta)^{\frac{p-2}{2}} \nabla u \cdot x |x|^{\alpha-2} dx \\ &= - \int_{\mathbb{R}^N} (|u|^2 + \delta)^{\frac{p-2}{2}} |\nabla u|^2 (1 + |x|^\alpha) dx \\ &\quad - (p-2) \int_{\mathbb{R}^N} (|u|^2 + \delta)^{\frac{p-4}{2}} (|u| \nabla |u|) \cdot (\bar{u} \nabla u) (1 + |x|^\alpha) dx \\ &\quad - \alpha \int_{\mathbb{R}^N} \bar{u}(|u|^2 + \delta)^{\frac{p-2}{2}} \nabla u \cdot x |x|^{\alpha-2} dx. \end{aligned}$$

So, using the identities $|\nabla |u||^2 \leq |\nabla u|^2$ and $|u| \nabla |u| = \text{Re}(\bar{u} \nabla u)$, we obtain

$$\begin{aligned} \text{Re} \langle Lu, u|u|^{p-2} \rangle &\leq -(p-1) \int_{\mathbb{R}^N} |\nabla |u||^2 |u|^{p-2} (1 + |x|^\alpha) dx \\ &\quad - \alpha \int_{\mathbb{R}^N} |u|^{p-1} \nabla |u| \cdot x |x|^{\alpha-2} dx \end{aligned}$$

if $p \geq 2$. The case $1 < p < 2$ can be handled similarly. Thus, by Hölder's inequality we have

$$\begin{aligned} \text{Re} \langle (L+V)u, u|u|^{p-2} \rangle &\leq -(p-1) \int_{\mathbb{R}^N} |\nabla |u||^2 |u|^{p-2} (1 + |x|^\alpha) dx + \int_{\mathbb{R}^N} V |u|^p dx \\ &\quad + \alpha \left(\int_{\mathbb{R}^N} |\nabla |u||^2 |u|^{p-2} |x|^\alpha dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} |x|^\alpha dx \right)^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Set

$$\begin{aligned} I_\alpha^2 &= \int_{\mathbb{R}^N} |\nabla |u||^2 |u|^{p-2} |x|^\alpha dx, & J_\alpha^2 &= \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} |x|^\alpha dx \\ I_0^2 &= \int_{\mathbb{R}^N} |\nabla |u||^2 |u|^{p-2} dx, & J_0^2 &= \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2} dx. \end{aligned}$$

Taking the assumption on V into account we obtain

$$\text{Re} \langle (L+V)u, u|u|^{p-2} \rangle \leq -(p-1)I_0^2 - (p-1)I_\alpha^2 + cJ_0^2 + \alpha I_\alpha J_\alpha.$$

Since $c \leq (p-1)\gamma_0$ and Lemma 2.4 holds for $\alpha = 0$, we have that $-(p-1)I_0^2 + cJ_0^2 \leq 0$. Now, the inequality

$$-(p-1)I_\alpha^2 + \alpha I_\alpha J_\alpha \leq 0$$

holds true if

$$-(p-1) + \alpha \gamma_\alpha^{-1/2} \leq 0,$$

thanks again to Lemma 2.4. The latter inequality is equivalent to $\alpha \leq (N-2)(p-1)$, which is the assumption. This ends the proof. \square

Remark 2.8. The assumption $\alpha \leq (N-2)(p-1)$ is optimal for the dissipativity of L , as proved in [11, Proposition 8.2].

Hence, in order to apply Theorem 1.1, we have established

Corollary 2.9. *Assume $\alpha \leq (N-2)(p-1)$. Then, the operator $L + \frac{c}{|x|^2}$ with $c \leq (p-1)\gamma_0$ and domain $C_c^\infty(\mathbb{R}^N)$ is dissipative in $L^p(\mathbb{R}^N)$.*

Let us recall the definition of dispersivity of an operator. A (real) linear operator A with domain $D(A)$ in $L^p(\mathbb{R}^N)$ is called dispersive if

$$\langle Au, u_+^{p-1} \rangle \leq 0 \quad \text{for all } u \in D(A).$$

For more details on dispersive operators we refer to [14, C-II.1].

Proposition 2.10. *Assume $\alpha \leq (N-2)(p-1)$. Then, the operator $L + \frac{c}{|x|^2}$ with $c \leq (p-1)\gamma_0$ and domain $C_c^\infty(\mathbb{R}^N)$ is dispersive in $L^p(\mathbb{R}^N)$.*

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$ be real-valued and fix $\delta > 0$. Replacing u by u_+ in the proof of Proposition 2.7 and since $u_+ \in W^{1,p}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned} \langle Lu, u_+(u_+^2 + \delta)^{\frac{p-2}{2}} \rangle &= - \int_{\mathbb{R}^N} (u_+^2 + \delta)^{\frac{p-2}{2}} |\nabla u_+|^2 (1 + |x|^\alpha) dx \\ &\quad - (p-2) \int_{\mathbb{R}^N} (u_+^2 + \delta)^{\frac{p-4}{2}} u_+^2 |\nabla u_+|^2 (1 + |x|^\alpha) dx \\ &\quad - \alpha \int_{\mathbb{R}^N} u_+ (u_+^2 + \delta)^{\frac{p-2}{2}} \nabla u_+ \cdot x |x|^{\alpha-2} dx. \end{aligned}$$

Then,

$$\begin{aligned} \langle Lu, u_+(u_+^2 + \delta)^{\frac{p-2}{2}} \rangle &\leq (1-p) \int_{\mathbb{R}^N} (u_+^2 + \delta)^{\frac{p-4}{2}} u_+^2 |\nabla u_+|^2 (1 + |x|^\alpha) dx \\ &\quad - \alpha \int_{\mathbb{R}^N} u_+ (u_+^2 + \delta)^{\frac{p-2}{2}} \nabla u_+ \cdot x |x|^{\alpha-2} dx, \end{aligned}$$

where here we take $\delta = 0$ if $p \geq 2$ and $\delta > 0$ if $1 < p < 2$. Thus, letting $\delta \rightarrow 0$ if $1 < p < 2$, and applying Hölder's inequality we obtain

$$\langle (L + \frac{c}{|x|^2})u, u_+^{p-1} \rangle \leq (1-p)I_{0,+}^2 + (1-p)I_{\alpha,+}^2 + cJ_{0,+}^2 + \alpha I_{\alpha,+} J_{\alpha,+},$$

where

$$\begin{aligned} I_{\alpha,+}^2 &= \int_{\mathbb{R}^N} |\nabla u_+|^2 u_+^{p-2} |x|^\alpha dx, & J_{\alpha,+}^2 &= \int_{\mathbb{R}^N} \frac{u_+^p}{|x|^2} |x|^\alpha dx \\ I_{0,+}^2 &= \int_{\mathbb{R}^N} |\nabla u_+|^2 u_+^{p-2} dx, & J_{0,+}^2 &= \int_{\mathbb{R}^N} \frac{u_+^p}{|x|^2} dx. \end{aligned}$$

As in the proof of Proposition 2.7, the assertion follows now by Lemma 2.4 and Remark 2.6. \square

The next proposition deals with the operator $L - \eta|x|^\beta$.

Proposition 2.11. *Let $V \in L_{loc}^p(\mathbb{R}^N \setminus \{0\})$ with $V \leq \frac{c}{|x|^2}$, $x \neq 0$ and $c \leq (p-1)\gamma_0$. Set $\tilde{L} = L - \eta|x|^\beta$.*

- (i) *If $\alpha \geq 2$, $\beta > \alpha - 2$ and $\eta > 0$ then the operator $\tilde{L} + V$ with domain $C_c^\infty(\mathbb{R}^N)$ is quasi-dissipative in $L^p(\mathbb{R}^N)$.*
- (ii) *If $0 \leq \alpha \leq (N-2)(p-1)$, $\beta = \alpha - 2$ then $\tilde{L} + V$ with domain $C_c^\infty(\mathbb{R}^N)$ is dissipative in $L^p(\mathbb{R}^N)$ if*

$$\eta + \frac{(N + \alpha - 2)^2}{pp'} - \frac{\alpha(N + \alpha - 2)}{p} \geq 0. \quad (2.4)$$

Proof. (i) If $\beta > \alpha - 2$, applying (2.3) and Young's inequality we obtain

$$\begin{aligned} \operatorname{Re}\langle (\tilde{L} + V)u, u|u|^{p-2} \rangle &\leq -(p-1)I_0^2 - (p-1)I_\alpha^2 + cJ_0^2 + \varepsilon I_\alpha^2 \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{\alpha^2}{4\varepsilon} |x|^{\alpha-2} - \eta |x|^\beta \right) |u|^p dx \\ &\leq -(p-1)I_0^2 - (p-1-\varepsilon)I_\alpha^2 + cJ_0^2 + M\|u\|_p^p \end{aligned}$$

for $u \in C_c^\infty(\mathbb{R}^N)$ and any $\varepsilon > 0$, where M is a positive constant such that $\frac{\alpha^2}{4\varepsilon} |x|^{\alpha-2} - \eta |x|^\beta \leq M$ for all $x \in \mathbb{R}^N$, which holds since $\beta > \alpha - 2 \geq 0$. Choosing now $\varepsilon \leq p-1$ and applying (2.2) we obtain

$$\operatorname{Re}\langle (\tilde{L} + V)u, u|u|^{p-2} \rangle \leq M\|u\|_p^p$$

which means that $\tilde{L} + V$ with domain $C_c^\infty(\mathbb{R}^N)$ is quasi-dissipative in $L^p(\mathbb{R}^N)$.

(ii) If $\beta = \alpha - 2$ then (2.3) gives

$$\operatorname{Re}\langle (\tilde{L} + V)u, u|u|^{p-2} \rangle \leq -(p-1)I_0^2 - (p-1)I_\alpha^2 + cJ_0^2 + \alpha I_\alpha J_\alpha - \eta J_\alpha^2.$$

If $\eta \geq 0$, then the conclusion easily follows as in the end of the proof of Proposition 2.7, under the assumption $c \leq \gamma_0(p-1)$ and $\alpha \leq (N-2)(p-1)$. If $\eta < 0$ then by Lemma 2.4 we have

$$-(p-1)I_\alpha^2 + \alpha I_\alpha J_\alpha - \eta J_\alpha^2 \leq (-(p-1) + \alpha\gamma_\alpha^{-1/2} - \eta\gamma_\alpha^{-1})I_\alpha^2.$$

The right hand side is nonpositive if

$$\eta + \frac{(N + \alpha - 2)^2}{pp'} - \frac{\alpha(N + \alpha - 2)}{p} \geq 0.$$

□

Remark 2.12. Condition (2.4) is sharp as proved in [12, Proposition 4.2].

3. MAIN RESULTS

In this section we state and prove the main results of this paper.

In order to apply Theorem 1.1 to our situation we need the following lemma whose proof follows the same lines of [16, Lemma 3.4].

Lemma 3.1. Set $V_\varepsilon = \frac{1}{|x|^{2+\varepsilon}}$, $\varepsilon > 0$. Assume $\alpha \leq (N-2)(p-1)$. Then for every $u \in C_c^\infty(\mathbb{R}^N)$

$$\operatorname{Re}\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx + \beta_\alpha \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx, \quad (3.1)$$

where

$$\beta_0 = \frac{N(p-1)(N-2p)}{p^2}, \quad \beta_\alpha = \frac{(Np - N - \alpha)(N + \alpha - 2p)}{p^2}.$$

Moreover, if $N > 2p$ then both β_0 and β_α are positive.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$ and set $u_\delta = ((R|u|)^2 + \delta)^{\frac{1}{2}}$, where $R^p := V_\varepsilon^{p-1}$. In the computations below, we have to take $\delta > 0$ in the case $1 < p < 2$, whereas we only take $\delta = 0$ to deal with the case $p \geq 2$. We have

$$\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle = - \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} Lu dx.$$

Integrating by parts we have

$$\begin{aligned}
-\int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} Lu \, dx &= \int_{\mathbb{R}^N} R^2 \bar{u} \nabla u \cdot \nabla (u_\delta^{p-2})(1 + |x|^\alpha) dx \\
&\quad + \int_{\mathbb{R}^N} u_\delta^{p-2} \nabla u \cdot \nabla (R^2 \bar{u})(1 + |x|^\alpha) dx \\
&\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} |x|^{\alpha-2} x \cdot \nabla u \, dx.
\end{aligned} \tag{3.2}$$

Now, computing $\nabla(u_\delta^{p-2})$ and writing $R^2 \bar{u} \nabla u = R \bar{u} (\nabla(Ru) - u \nabla R)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} R^2 \bar{u} \nabla u \cdot \nabla (u_\delta^{p-2})(1 + |x|^\alpha) dx \\
&= \frac{p-2}{2} \int_{\mathbb{R}^N} u_\delta^{p-4} R \bar{u} \nabla(R^2 |u|^2) \cdot \nabla(Ru)(1 + |x|^\alpha) dx \\
&\quad - \frac{p-2}{2} \int_{\mathbb{R}^N} u_\delta^{p-4} R |u|^2 \nabla(R^2 |u|^2) \cdot \nabla R (1 + |x|^\alpha) dx.
\end{aligned}$$

Using also the identity

$$\nabla(R^2 \bar{u}) \cdot \nabla u = |\nabla(Ru)|^2 - u \nabla(R \bar{u}) \cdot \nabla R + R \bar{u} \nabla R \cdot \nabla u$$

Equation (3.2) yields

$$\begin{aligned}
-\int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} Lu \, dx &= \frac{p-2}{2} \int_{\mathbb{R}^N} u_\delta^{p-4} R \bar{u} \nabla(R^2 |u|^2) \cdot \nabla(Ru)(1 + |x|^\alpha) dx \\
&\quad + \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(Ru)|^2 (1 + |x|^\alpha) dx \\
&\quad - \underbrace{\frac{p-2}{2} \int_{\mathbb{R}^N} u_\delta^{p-4} R |u|^2 \nabla(R^2 |u|^2) \cdot \nabla R (1 + |x|^\alpha) dx}_{=I} \\
&\quad + \underbrace{\int_{\mathbb{R}^N} u_\delta^{p-2} R \bar{u} \nabla R \cdot \nabla u (1 + |x|^\alpha) dx}_{=J} \\
&\quad - \underbrace{\int_{\mathbb{R}^N} u_\delta^{p-2} u \nabla(R \bar{u}) \cdot \nabla R (1 + |x|^\alpha) dx}_{=K} \\
&\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} |x|^{\alpha-2} x \cdot \nabla u \, dx.
\end{aligned}$$

Now, introduce the function $Q = R^p$. Writing $\nabla(R^2 |u|^2) = 2R |u|^2 \nabla R + 2|u| R^2 \nabla |u|$ we have

$$\begin{aligned}
I &= -(p-2) \int_{\mathbb{R}^N} u_\delta^{p-4} R^2 |u|^4 |\nabla R|^2 (1 + |x|^\alpha) dx \\
&\quad - (p-2) \int_{\mathbb{R}^N} u_\delta^{p-4} |u|^3 R^3 \nabla R \cdot \nabla |u| (1 + |x|^\alpha) dx \\
&= -\frac{p-2}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-4} R^{4-2p} |u|^4 |\nabla Q|^2 (1 + |x|^\alpha) dx \\
&\quad - \frac{p-2}{p} \int_{\mathbb{R}^N} u_\delta^{p-4} |u|^3 R^{4-p} \nabla Q \cdot \nabla |u| (1 + |x|^\alpha) dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
J + K &= \int_{\mathbb{R}^N} u_\delta^{p-2} \left(R \bar{u} \nabla R \cdot \nabla u - u \nabla(R \bar{u}) \cdot \nabla R \right) (1 + |x|^\alpha) dx \\
&= - \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 |\nabla R|^2 (1 + |x|^\alpha) dx \\
&\quad + 2i \int_{\mathbb{R}^N} u_\delta^{p-2} \mathcal{I}m(\bar{u} \nabla u) \cdot R \nabla R (1 + |x|^\alpha) dx \\
&= - \frac{1}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 R^{2-2p} |\nabla Q|^2 (1 + |x|^\alpha) dx \\
&\quad + \frac{2i}{p} \int_{\mathbb{R}^N} u_\delta^{p-2} R^{2-p} \mathcal{I}m(\bar{u} \nabla u) \cdot \nabla Q (1 + |x|^\alpha) dx.
\end{aligned}$$

Hence we have

$$\begin{aligned}
- \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} Lu dx &= (p-2) \int_{\mathbb{R}^N} u_\delta^{p-4} R |u| \nabla(R|u|) \cdot (R \bar{u}) \nabla(Ru) (1 + |x|^\alpha) dx \\
&\quad + \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(Ru)|^2 (1 + |x|^\alpha) dx + J_\delta \\
&\quad + \frac{2i}{p} \int_{\mathbb{R}^N} u_\delta^{p-2} R^{2-p} \mathcal{I}m(\bar{u} \nabla u) \cdot \nabla Q (1 + |x|^\alpha) dx \\
&\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} |x|^{\alpha-2} x \cdot \nabla u dx,
\end{aligned}$$

where we have set

$$\begin{aligned}
J_\delta &= - \frac{p-2}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-4} R^{4-2p} |u|^4 |\nabla Q|^2 (1 + |x|^\alpha) dx \\
&\quad - \frac{p-2}{p} \int_{\mathbb{R}^N} u_\delta^{p-4} |u|^3 R^{4-p} \nabla Q \cdot \nabla |u| (1 + |x|^\alpha) dx \\
&\quad - \frac{1}{p^2} \int_{\mathbb{R}^N} u_\delta^{p-2} |u|^2 R^{2-2p} |\nabla Q|^2 (1 + |x|^\alpha) dx.
\end{aligned}$$

Now, we take the real parts of both sides and apply the identity $\mathcal{R}e(\bar{\phi} \nabla \phi) = |\phi| \nabla |\phi|$ to obtain

$$\begin{aligned}
-\mathcal{R}e \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} Lu dx &= (p-2) \int_{\mathbb{R}^N} u_\delta^{p-4} R^2 |u|^2 |\nabla(R|u|)|^2 (1 + |x|^\alpha) dx \\
&\quad + \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(Ru)|^2 (1 + |x|^\alpha) dx + J_\delta \\
&\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 |u| |x|^{\alpha-2} x \cdot \nabla |u| dx \\
&= (p-2) \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(R|u|)|^2 (1 + |x|^\alpha) dx \\
&\quad - (p-2) \delta \int_{\mathbb{R}^N} u_\delta^{p-4} |\nabla(R|u|)|^2 (1 + |x|^\alpha) dx \\
&\quad + \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(Ru)|^2 (1 + |x|^\alpha) dx + J_\delta \\
&\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 |u| |x|^{\alpha-2} x \cdot \nabla |u| dx.
\end{aligned}$$

Since the inequality $|\nabla\phi| \geq |\nabla|\phi||$ holds and $\delta = 0$ if $p \geq 2$, $\delta > 0$ if $1 < p < 2$ we can estimate as follows

$$\begin{aligned} -\mathcal{R}e \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} L u \, dx &\geq (p-1) \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(R|u|)|^2 (1 + |x|^\alpha) \, dx + J_\delta \\ &\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 |u| |x|^{\alpha-2} x \cdot \nabla |u| \, dx \end{aligned}$$

if $p \geq 2$ and

$$\begin{aligned} -\mathcal{R}e \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 \bar{u} L u \, dx &\geq (p-1) \int_{\mathbb{R}^N} u_\delta^{p-2} |\nabla(Ru)|^2 (1 + |x|^\alpha) \, dx + J_\delta \\ &\quad + \alpha \int_{\mathbb{R}^N} u_\delta^{p-2} R^2 |u| |x|^{\alpha-2} x \cdot \nabla |u| \, dx \end{aligned}$$

if $1 < p < 2$. Letting $\delta \rightarrow 0^+$, we are lead to

$$\begin{aligned} \mathcal{R}e \langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle &\geq (p-1) \int_{\mathbb{R}^N} (R|u|)^{p-2} |\nabla(R|u|)|^2 (1 + |x|^\alpha) \, dx \\ &\quad - \frac{p-1}{p^2} \int_{\mathbb{R}^N} R^{-p} |u|^p |\nabla Q|^2 (1 + |x|^\alpha) \, dx \\ &\quad - \frac{p-2}{p} \int_{\mathbb{R}^N} |u|^{p-1} \nabla Q \cdot \nabla |u| (1 + |x|^\alpha) \, dx \\ &\quad + \alpha \int_{\mathbb{R}^N} |u|^{p-1} R^p |x|^{\alpha-2} x \cdot \nabla |u| \, dx, \end{aligned} \tag{3.3}$$

where we have used again the inequality $|\nabla\phi| \geq |\nabla|\phi||$ in the first integral of the right hand side of (3.3), since for $1 < p < 2$ we had $|\nabla Ru|^2$ instead of $|\nabla(R|u|)|^2$. Now, by the identity $p|u|^{p-1} \nabla |u| = \nabla |u|^p$, integrating by parts and recalling the definition of R we infer

$$\begin{aligned} & - \frac{p-2}{p} \int_{\mathbb{R}^N} |u|^{p-1} \nabla Q \cdot \nabla |u| (1 + |x|^\alpha) \, dx \\ &= \frac{p-2}{p^2} \int_{\mathbb{R}^N} |u|^p \Delta R^p (1 + |x|^\alpha) \, dx + \frac{\alpha(p-2)}{p^2} \int_{\mathbb{R}^N} |u|^p \nabla R^p \cdot x |x|^{\alpha-2} \, dx \\ &= - \frac{2N(p-1)(p-2)}{p^2} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p (1 + |x|^\alpha) \, dx \\ &\quad + \frac{4p(p-1)(p-2)}{p^2} \int_{\mathbb{R}^N} |x|^2 V_\varepsilon^{p+1} |u|^p (1 + |x|^\alpha) \, dx \\ &\quad - \frac{2\alpha(p-1)(p-2)}{p^2} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha \, dx \end{aligned}$$

and

$$\begin{aligned} & \alpha \int_{\mathbb{R}^N} |u|^{p-1} R^p |x|^{\alpha-2} x \cdot \nabla |u| \, dx \\ &= - \frac{\alpha}{p} \int_{\mathbb{R}^N} |u|^p |x|^{\alpha-2} x \cdot \nabla R^p \, dx - \frac{\alpha(N+\alpha-2)}{p} \int_{\mathbb{R}^N} R^p |x|^{\alpha-2} |u|^p \, dx \\ &= \frac{2(p-1)\alpha}{p} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha \, dx - \frac{\alpha(N+\alpha-2)}{p} \int_{\mathbb{R}^N} V_\varepsilon^{p-1} |u|^p |x|^{\alpha-2} \, dx. \end{aligned}$$

Finally,

$$\int_{\mathbb{R}^N} |u|^p R^{-p} |\nabla R^p|^2 (1 + |x|^\alpha) \, dx = 4(p-1)^2 \int_{\mathbb{R}^N} |x|^2 V_\varepsilon^{p+1} |u|^p (1 + |x|^\alpha) \, dx.$$

By using such formulas in (3.3) we obtain

$$\begin{aligned} \operatorname{Re}\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle &\geq (p-1) \int_{\mathbb{R}^N} (R|u|)^{p-2} |\nabla(R|u|)|^2 (1 + |x|^\alpha) dx \\ &\quad - \frac{4(p-1)}{p^2} \int_{\mathbb{R}^N} |x|^2 V_\varepsilon^{p+1} |u|^p (1 + |x|^\alpha) dx \\ &\quad - \frac{2N(p-1)(p-2)}{p^2} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p (1 + |x|^\alpha) dx \\ &\quad + \frac{4\alpha(p-1)}{p^2} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx \\ &\quad - \frac{\alpha(N+\alpha-2)}{p} \int_{\mathbb{R}^N} V_\varepsilon^{p-1} \frac{|u|^p}{|x|^2} |x|^\alpha dx. \end{aligned}$$

Applying Lemma 2.4, and using that $|x|^2 V_\varepsilon \leq 1$ we are lead to

$$\begin{aligned} \operatorname{Re}\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle &\geq (p-1)\gamma_0 \int_{\mathbb{R}^N} \frac{V_\varepsilon^{p-1} |u|^p}{|x|^2} dx \\ &\quad + \frac{N+\alpha-2}{p} \left(\frac{p-1}{p} (N+\alpha-2) - \alpha \right) \int_{\mathbb{R}^N} \frac{V_\varepsilon^{p-1} |u|^p}{|x|^2} |x|^\alpha dx \\ &\quad - \frac{p-1}{p^2} (4+2Np-4N) \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p (1 + |x|^\alpha) dx \\ &\quad + \frac{4\alpha(p-1)}{p^2} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx. \end{aligned} \tag{3.4}$$

Since $\alpha \leq (N-2)(p-1)$ we have $\frac{p-1}{p} (N+\alpha-2) - \alpha \geq 0$ and then from the estimate $|x|^2 V_\varepsilon \leq 1$ it follows that

$$\begin{aligned} \operatorname{Re}\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle &\geq (p-1)\gamma_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx \\ &\quad + \frac{N+\alpha-2}{p} \left(\frac{p-1}{p} (N+\alpha-2) - \alpha \right) \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx \\ &\quad - \frac{p-1}{p^2} (4+2Np-4N) \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p (1 + |x|^\alpha) dx \\ &\quad + \frac{4\alpha(p-1)}{p^2} \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx. \end{aligned}$$

Thus we have

$$\operatorname{Re}\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx + \beta_\alpha \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx,$$

where

$$\begin{aligned} \beta_0 &= (p-1)\gamma_0 - \frac{p-1}{p^2} (4+2Np-4N) = \frac{N(p-1)(N-2p)}{p^2} \\ \beta_\alpha &= \frac{N+\alpha-2}{p} \left(\frac{p-1}{p} (N+\alpha-2) - \alpha \right) - \frac{p-1}{p^2} (4+2Np-4N) + \frac{4\alpha(p-1)}{p^2} \\ &= \frac{(Np-N-\alpha)(N+\alpha-2p)}{p^2}. \end{aligned}$$

So, if $N > 2p$ then $\beta_0 > 0$ and since $0 \leq \alpha \leq (N-2)(p-1) < N(p-1)$ we deduce that $\beta_\alpha > 0$. \square

Remark 3.2. We rewrite estimate (3.4) as follows

$$\operatorname{Re}\langle -Lu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx + \int_{\mathbb{R}^N} (k_0 + k_1 V_\varepsilon |x|^2) V_\varepsilon^{p-1} |u|^p |x|^{\alpha-2} dx$$

where

$$k_0 = \frac{N + \alpha - 2}{p} \left(\frac{p-1}{p} (N + \alpha - 2) - \alpha \right)$$

$$k_1 = \frac{4\alpha(p-1)}{p^2} - \frac{p-1}{p^2} (4 + 2Np - 4N).$$

Notice that $k_0 \geq 0$ if $\alpha \leq (N-2)(p-1)$ and that $k_0 + k_1 = \beta_\alpha$. Now, $k_0 + k_1 V_\varepsilon |x|^2 = f(|x|^2)$, where $f(r) = \frac{\varepsilon k_0 + (k_0 + k_1)r}{\varepsilon + r}$. Since $\inf_{[0, \infty)} f = \min\{k_0, k_0 + k_1\} =: \mu$ we find

$$\mathcal{R}e \langle -Lu - \mu |x|^{\alpha-2} u, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx.$$

The easiest case (see Lemma 3.1) is when $\mu \geq 0$.

Now, we prove a similar estimate for the operator $\tilde{L} = L - \eta |x|^\beta$.

Lemma 3.3. *Set $V_\varepsilon = \frac{1}{|x|^{2+\varepsilon}}$, $\varepsilon > 0$. If $\beta > \alpha - 2 \geq 0$ and $\eta > 0$ then for every $u \in C_c^\infty(\mathbb{R}^N)$*

$$\mathcal{R}e \langle -\tilde{L}u - mu, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx + \delta_\alpha \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx,$$

where $m = \min_{x \in \mathbb{R}^N} \left(\frac{N+\alpha-2}{p} \cdot \frac{(p-1)(N-2)-\alpha}{p} |x|^{\alpha-2} + \eta |x|^\beta \right)$, β_0 is given in Lemma 3.1 and

$$\delta_\alpha = \frac{p-1}{p^2} (4\alpha - 4 - 2Np + 4N).$$

Proof. We proceed as in the proof of Lemma 3.1. From Remark 3.2 and the inequality $|x|^2 V_\varepsilon \leq 1$ it follows that

$$\begin{aligned} & \mathcal{R}e \langle -\tilde{L}u, |V_\varepsilon u|^{p-2} V_\varepsilon u \rangle \\ & \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx \\ & \quad + \int_{\mathbb{R}^N} V_\varepsilon^{p-1} |u|^p \left(\frac{N+\alpha-2}{p} \cdot \frac{(p-1)(N-2)-\alpha}{p} |x|^{\alpha-2} + \eta |x|^\beta \right) dx \\ & \quad + \left(\frac{4\alpha(p-1)}{p^2} - \frac{p-1}{p^2} (4 + 2Np - 4N) \right) \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx \\ & \geq \beta_0 \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p dx + m \int_{\mathbb{R}^N} V_\varepsilon^{p-1} |u|^p dx + \delta_\alpha \int_{\mathbb{R}^N} V_\varepsilon^p |u|^p |x|^\alpha dx. \end{aligned}$$

Thus the proof of the lemma is concluded. \square

Applying Corollary 2.9, Lemma 3.1 and Theorem 1.1 we obtain the following generation results. We distinguish the two cases $\alpha \leq 2$ and $\alpha > 2$ since the hypotheses on the unperturbed operator L are different.

Theorem 3.4. *Assume $0 \leq \alpha \leq 2$. Set $k = \min\{\beta_0, (p-1)\gamma_0\}$. If $2p < N$ and $\alpha \leq (N-2)(p-1)$ then, for every $c < k$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain D_p defined in Theorem 2.1 generates a contractive positive C_0 -semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(L + \frac{k}{|x|^2}, D_p\right)$ generates a contractive positive C_0 -semigroup in $L^p(\mathbb{R}^N)$.*

Theorem 3.5. *Assume $\alpha > 2$. Set $k = \min\{\beta_0, (p-1)\gamma_0\}$. If $\frac{N}{N-2} < p < \frac{N}{2}$ and $\alpha < \frac{N(p-1)}{p}$, then for every $c < k$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain \widehat{D}_p given in Theorem 2.2 generates a contractive positive C_0 -semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(L + \frac{k}{|x|^2}, \widehat{D}_p\right)$ generates a contractive positive C_0 -semigroup in $L^p(\mathbb{R}^N)$.*

The proofs of the two above theorems are identical. We limit ourselves in proving the latter.

Proof of Theorem 3.5. In order to apply Theorem 1.1, set $A = -L$, $D(A) = \widehat{D}_p$, $D = C_c^\infty(\mathbb{R}^N)$ and let B be the multiplicative operator by $\frac{1}{|x|^2}$ endowed with the maximal domain $D(|x|^{-2}) = \{u \in L^p(\mathbb{R}^N); |x|^{-2}u \in L^p(\mathbb{R}^N)\}$ in $L^p(\mathbb{R}^N)$. We observe that the Yosida approximation B_ε of B is the multiplicative operator by $V_\varepsilon = \frac{1}{|x|^2 + \varepsilon}$. Both A and B are m -accretive in $L^p(\mathbb{R}^N)$. Then, Lemma 3.1 yields (i) in Theorem 1.1 with $k_1 = \beta_0$, $\tilde{c} = 0$ and $a = 0$. The second assumption (ii) in Theorem 1.1 is obviously satisfied. The last one, (iii), holds with $k_2 = (p-1)\gamma_0$ thanks to Corollary 2.9. Then, we infer that for every $c < k$, $-L - \frac{c}{|x|^2}$ with domain \widehat{D}_p is m -accretive in $L^p(\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}^N)$ is a core for $-L - \frac{c}{|x|^2}$ by Theorem 2.2. Moreover, $-L - \frac{k}{|x|^2}$ is essentially m -accretive. By the Lumer Phillips Theorem (cf. [5, Chap.II, Theorem 3.15]) we obtain the generation result. Finally, the positivity of the semigroup is a consequence of Proposition 2.10. The dispersivity is equivalent to the positivity of the resolvent, which is equivalent to the positivity of the semigroup. \square

If $2p \geq N$, then $\beta_0 \leq 0$ and we cannot apply Theorem 1.1. However, if at least $\beta_\alpha \geq 0$, that is $2p - N \leq \alpha$, then we still have a generation result, relying on the following abstract theorem by Okazawa (see [16, Theorem 1.6]).

Theorem 3.6. *Let A and B be linear m -accretive operators in $L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Let D be a core of A . Assume that there are constants $\tilde{c}, a, b \geq 0$ such that for all $u \in D$ and $\varepsilon > 0$,*

$$\operatorname{Re}\langle Au, \|B_\varepsilon u\|_p^{2-p} |B_\varepsilon u|^{p-2} B_\varepsilon u \rangle \geq -b \|B_\varepsilon u\|_p^2 - \tilde{c} \|u\|_p^2 - a \|B_\varepsilon u\|_p \|u\|_p,$$

where $B_\varepsilon := B(I + \varepsilon B)^{-1}$ denotes the Yosida approximation of B . If $\nu > b$ then $A + \nu B$ with domain $D(A) \cap D(B)$ is m -accretive and $D(A) \cap D(B)$ is core for A . Moreover, $A + bB$ is essentially m -accretive on $D(A) \cap D(B)$.

In our framework the above result leads to the following theorems. We recall that $D(|x|^{-2}) = \{u \in L^p(\mathbb{R}^N); |x|^{-2}u \in L^p(\mathbb{R}^N)\}$.

Theorem 3.7. *Assume $0 \leq \alpha \leq 2$. If $2p \geq N$ and $2p - N \leq \alpha \leq (N-2)(p-1)$ then, for every $c < \beta_0$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain $D_p \cap D(|x|^{-2})$, where D_p is defined in Theorem 2.1, generates a contractive analytic C_0 -semigroup in $L^p(\mathbb{R}^N)$. Moreover, the closure of $\left(L + \frac{\beta_0}{|x|^2}, D_p \cap D(|x|^{-2})\right)$ generates a contractive analytic C_0 -semigroup in $L^p(\mathbb{R}^N)$.*

Theorem 3.8. *Assume $\alpha > 2$. If $2p \geq N$ and $2p - N \leq \alpha < \frac{N(p-1)}{p}$, then for every $c < \beta_0$ the operator $L + \frac{c}{|x|^2}$ endowed with the domain $\widehat{D}_p \cap D(|x|^{-2})$, where \widehat{D}_p is given in Theorem 2.2, generates a contractive analytic C_0 -semigroup in $L^p(\mathbb{R}^N)$. Moreover, the closure of $\left(L + \frac{\beta_0}{|x|^2}, \widehat{D}_p \cap D(|x|^{-2})\right)$ generates a contractive analytic C_0 -semigroup in $L^p(\mathbb{R}^N)$.*

As before, we limit ourselves in proving the latter.

Proof of Theorem 3.8. In order to apply Theorem 3.6, set $A = -L$, $D(A) = \widehat{D}_p$, $D = C_c^\infty(\mathbb{R}^N)$ and let B be the multiplicative operator by $\frac{1}{|x|^2}$ endowed with the maximal domain $D(|x|^{-2})$ in $L^p(\mathbb{R}^N)$. Both A and B are m -accretive in $L^p(\mathbb{R}^N)$. Then, Lemma 3.1 and Theorem 3.6 (with $b = -\beta_0$, $\tilde{c} = 0$ and $a = 0$) imply that $\left(L + \frac{c}{|x|^2}, \widehat{D}_p \cap D(|x|^{-2})\right)$ is m -accretive in $L^p(\mathbb{R}^N)$ for any $c < \beta_0$ and is

essentially m -accretive if $c = \beta_0$. From the assumptions $2 < \alpha < \frac{N(p-1)}{p}$ it follows that $p > N/(N-2)$ and this yields $\alpha < (N-2)(p-1)$. Therefore, by Theorem 2.2, L generates a positive C_0 -semigroup of contractions, which is also analytic. By inspecting the proof of [11, Theorem 8.1] it turns out that there exists $\ell_\alpha > 0$ such that

$$|\operatorname{Im}\langle Lu, |u|^{p-2}u \rangle| \leq \ell_\alpha (-\operatorname{Re}\langle Lu, |u|^{p-2}u \rangle)$$

for every $u \in \widehat{D_p}$ (the computations can be performed for $u \in C_c^\infty(\mathbb{R}^N)$ and then one get the estimate for $u \in \widehat{D_p}$ using the fact that $C_c^\infty(\mathbb{R}^N)$ is a core for L). Now, the previous estimate continues to hold for all $u \in \widehat{D_p} \cap D(|x|^{-2})$ replacing L with $L + \frac{c}{|x|^2}$, $c \leq \beta_0$. This implies that $e^{\pm i\theta} \left(L + \frac{c}{|x|^2} \right)$ is dissipative, where $\cot \theta = \ell_\alpha$. By [5, Theorem 4.6, Chapter 2], it follows that $L + \frac{c}{|x|^2}$ is sectorial and hence generates an analytic semigroup in $L^p(\mathbb{R}^N)$. This ends the proof. \square

If we consider the operator \widetilde{L} instead of L the above conditions on p can be simplified. So, by Theorem 2.3, Proposition 2.11 and Lemma 3.3, we can apply Theorem 1.1 (Theorem 3.6, respectively) since $\delta_\alpha \geq 0$ if and only if $\alpha \geq 1 + \frac{N}{2}(p-2)$.

Theorem 3.9. *Assume $\beta > \alpha - 2 > 0$ and $\eta > 0$. Set $k = \min\{\beta_0, (p-1)\gamma_0\}$. If $\alpha \geq 1 + \frac{N}{2}(p-2)$ and $N > 2p$ then for every $c < k$, the operator $\widetilde{L} + \frac{c}{|x|^2}$ endowed with the domain $\widetilde{D_p}$ given in Theorem 2.3 generates a positive and quasi-contractive C_0 -semigroup in $L^p(\mathbb{R}^N)$. Moreover, $C_c^\infty(\mathbb{R}^N)$ is a core for such an operator. Finally, the closure of $\left(\widetilde{L} + \frac{k}{|x|^2}, \widetilde{D_p} \right)$ generates a positive and quasi-contractive C_0 -semigroup in $L^p(\mathbb{R}^N)$.*

Theorem 3.10. *Assume $\beta > \alpha - 2 > 0$ and $\eta > 0$. If $\alpha \geq 1 + \frac{N}{2}(p-2)$ and $N \leq 2p$ then for every $c < \beta_0$, the operator $\widetilde{L} + \frac{c}{|x|^2}$ endowed with the domain $\widetilde{D_p} \cap D(|x|^{-2})$ generates a quasi-contractive C_0 -semigroup in $L^p(\mathbb{R}^N)$. Moreover, the closure of $\left(\widetilde{L} + \frac{\beta_0}{|x|^2}, \widetilde{D_p} \cap D(|x|^{-2}) \right)$ generates a quasi-contractive C_0 -semigroup in $L^p(\mathbb{R}^N)$.*

Let us end with the study of the optimality of the constant β_0 in (3.1).

Proposition 3.11. *Assume that*

$$\operatorname{Re}\langle -Lu, |Vu|^{p-2}Vu \rangle \geq C\|Vu\|_p^p, \quad (3.5)$$

for some $C > 0$, where $V = \frac{1}{|x|^2}$ and $\alpha \in \mathbb{N}$. Then, $C \leq \beta_0$.

Proof. Take $u(x) = v(r) \geq 0$, $r = |x|$. Then

$$\begin{aligned} \operatorname{Re}\langle -Lu, |Vu|^{p-2}Vu \rangle &= -\omega_N \int_0^{+\infty} (1+r^\alpha) \left(v'' + \frac{N-1}{r}v' \right) r^{-2(p-1)}v^{p-1}r^{N-1}dr \\ &= J, \end{aligned}$$

where ω_N denotes the measure of the unit ball in \mathbb{R}^N . Choose $v(r) = r^\beta e^{-r/p}$, with $\beta > \frac{2p-N}{p}$. Then

$$J = -\omega_N \int_0^{+\infty} (1+r^\alpha) \left(\beta(\beta+N-2)r^{\delta-1} + \frac{1-N-2\beta}{p}r^\delta + \frac{1}{p^2}r^{\delta+1} \right) e^{-r}dr,$$

where we have set $\delta = \beta p + N - 2p$. Notice that $\delta > 0$ thanks to the choice of β . Using the properties of the Euler Gamma function, we have

$$\begin{aligned} J &= -\omega_N \left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}\delta + \frac{1}{p^2}\delta(\delta+1) \right) \Gamma(\delta) \\ &\quad - \omega_N \left(\beta(\beta+N-2) + \frac{1-N-2\beta}{p}(\delta+\alpha) + \frac{1}{p^2}(\delta+\alpha)(\delta+\alpha+1) \right) \Gamma(\delta+\alpha). \end{aligned}$$

Now, observe that $\|Vu\|_p^p = \omega_N \Gamma(\delta)$. Hence from (3.5) it follows that

$$C \Gamma(\delta) \leq - \left(\beta(\beta + N - 2) + \frac{1 - N - 2\beta}{p} \delta + \frac{1}{p^2} \delta(\delta + 1) \right) \Gamma(\delta) \\ - \left(\beta(\beta + N - 2) + \frac{1 - N - 2\beta}{p} (\delta + \alpha) + \frac{1}{p^2} (\delta + \alpha)(\delta + \alpha + 1) \right) \Gamma(\delta + \alpha).$$

If $\alpha = n \in \mathbb{N}$ then $\Gamma(\delta + n) = (\delta + n - 1) \cdots \delta \Gamma(\delta)$ and the previous estimate yields

$$C \leq - \left(\beta(\beta + N - 2) + \frac{1 - N - 2\beta}{p} \delta + \frac{1}{p^2} \delta(\delta + 1) \right) \\ - \left(\beta(\beta + N - 2) + \frac{1 - N - 2\beta}{p} (\delta + n) + \frac{1}{p^2} (\delta + n)(\delta + n + 1) \right) (\delta + n - 1) \cdots \delta.$$

Letting $\delta \rightarrow 0^+$ which corresponds to $\beta \rightarrow \frac{2p-N}{p}$ eventually implies

$$C \leq \frac{N(p-1)(N-2p)}{p^2}.$$

Hence β_0 is the best constant for (3.5) to hold in the case $\alpha \in \mathbb{N}$. \square

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